THE COHOMOLOGY OF ALGEBRAS OVER MODULI SPACES

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The purpose of this paper is to introduce the cohomology of various algebras over an operad of moduli spaces including the cohomology of conformal field theories (CFT's) and vertex operator algebras (VOA's). This cohomology theory produces a number of invariants of CFT's and VOA's, one of which is the space of their infinitesimal deformations.

The paper is inspired by the ideas of Drinfeld [5], Kontsevich [16] and Ginzburg and Kapranov [10] on Koszul duality for operads. An operad is a gadget which parameterizes algebraic operations on a vector space. They were originally invented [21] in order to study the homotopy type of iterated loop spaces. Recently, operads have turned out to be an effective tool in describing various algebraic structures that arise in mathematical physics in terms of the geometry of moduli spaces particularly in conformal field theory (see [8], [12], [14], [19], [26], [27]) and topological gravity (see [15], [18]). In fact, a (tree level $c = 0$) conformal field theory is nothing more than a representation of the operad, $\mathcal{P}$, consisting of moduli spaces of configurations of holomorphically embedded unit disks in the Riemann sphere. Alternately, a conformal field theory is said to be a $\mathcal{P}$-algebra or an algebra over $\mathcal{P}$.

In the first part of this paper, we use the idea of Ginzburg-Kapranov [10] of homology of an algebra over a quadratic operad to define the cohomology of an algebra over a quadratic operad with values in an arbitrary module. We prove that $\mathcal{P}$ is a quadratic operad and construct its Koszul dual operad. We then construct the cohomology theory associated to a conformal field theory. We demonstrate that the second cohomology group, as it should, parameterizes deformations of conformal field theories. Our approach to deformations is morally the same as the one developed by Dijkgraaf and E. and H. Verlinde [3, 4] and Ranganathan, Sonoda and Zwiebach [22] using (1,1)-fields, except that we fix the action of the Virasoro algebra. It would be very interesting to find an explicit connection between the two approaches. By analogy with the case of associative algebras, one expects the third cohomology group to contain obstructions to extending an infinitesimal deformation to a formal neighborhood. Another plausible interpretation of higher dimensional cohomology is that

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they should parameterize infinitesimal deformations inside a larger category of "A_∞-conformal field theories" where one allows multilinear vertex operators which are not compositions of bilinear ones.

The second part of this paper is structured in the opposite way. We construct the cohomology theory associated to algebras over the little intervals operad, $\mathcal{B}$ – the space of configurations of intervals embedded in the interval via translations and dilations – by studying deformations of $\mathcal{B}$-algebras in analogy with the manner in which Hochschild cohomology arises from deformations of associative algebras. We do so by realizing associative algebras as one dimensional topological field theories. The little intervals operad is an important suboperad of the real analytic analog of $\mathcal{P}$ which can be regarded as a suboperad of $\mathcal{P}$. Therefore, a conformal field theory is a $\mathcal{B}$-algebra. This leads us to a complex which, in the case where the conformal field theory is a so-called vertex operator algebra, can be written explicitly. This complex closely resembles Hochschild cohomology and looks like what one would obtain from formally deforming the operator product in a vertex operator algebra.

Throughout the paper, we restrict our attention to tree level theories, those which correspond to Riemann surfaces of genus zero. There should be a cyclic version of the cohomology theory considered here, as it happens for the homology of algebras over cyclic operads, see [9].

A computation of the cohomology of a conformal field theory would provide valuable information about the structure of the moduli space of conformal field theories which plays an important role in related phenomena such as mirror symmetry, cf. Kontsevich and Manin [18]. We expect techniques from the theory of vertex operator algebras to be useful in performing this computation.

1. Algebras and operads. Assume for simplicity that all vector spaces are over $\mathbb{C}$. Perhaps the simplest example of an operad is the *endomorphism operad* of a vector space V, denoted by $\mathcal{E}nd_V = \{\mathcal{E}nd_V(n)\}_{n \geq 1}$, which is a collection of spaces $\mathcal{E}nd_V(n) = \mathrm{Hom}(V^{\otimes n}, V)$, each with the natural action of the permutation group, S_n, which acts by permuting the factors of the tensor product, and which have natural compositions between them, namely, the composition, $f \circ_i f'$, of two elements f in $\mathcal{O}(n)$ and f' in $\mathcal{O}(n')$ is obtained by

$$(f \circ_i f')(v_1 \otimes \cdots \otimes v_{n+n'-1})$$
$$= f(v_1, \otimes \cdots \otimes v_{i-1} \otimes f'(v_i \otimes \cdots v_{i+n'-1}) \otimes \cdots \otimes v_{n+n'-1})$$

for all $i = 1, \ldots, n$, and the permutation groups act equivariantly with respect to the compositions. Finally, there is an element I in $\mathcal{E}nd_V(1)$, the identity map $I : V \to V$, which is a unit with respect to the composition maps. This structure can be formalised in the following way.

An *operad* $\mathcal{O} = \{\mathcal{O}(n)\}_{n \geq 1}$ *with unit* is a collection of objects (topological spaces, vector spaces, etc. – elements of any symmetric monoidal category) such that each

$\mathcal{O}(n)$ has an action of S_n, the permutation group on n elements, and a collection of operations for $n \geq 1$ and $1 \leq i \leq n$, $\mathcal{O}(n) \times \mathcal{O}(n') \rightarrow \mathcal{O}(n + n' - 1)$ given by $(f, f') \mapsto f \circ_i f'$ such that

(1) if $f \in \mathcal{O}(n)$, $f' \in \mathcal{O}(n')$, and $f'' \in \mathcal{O}(n'')$ where $1 \leq i < j \leq n$ then

$$(f \circ_i f') \circ_{j+n'-1} f'' = (f \circ_j f'') \circ_i f'$$

(2) if $f \in \mathcal{O}(n)$, $f' \in \mathcal{O}(n')$, and $f'' \in \mathcal{O}(n'')$ where $n, n' \geq 1$ and $i = 1, \ldots, n$ and $j = 1, \ldots, n'$ then

$$(f \circ_i f') \circ_{i+j-1} f'' = f \circ_i (f' \circ_j f''),$$

(3) the composition maps are equivariant under the action of the permutation groups,
(4) there exists an element I in $\mathcal{O}(1)$ called the *unit* such that for all f in $\mathcal{O}(n)$ and $i = 1, \ldots, n$,

$$I \circ_1 f = f = f \circ_i I$$

This definition of an operad can be generalized by including $\mathcal{O}(0)$ or by relaxing the condition that the unit element exist but at the cost of having to introduce additional axioms. We shall see that many of the constructions which follow are quite at ease in this more general setting.

Given an operad, there is a notion of a representation of this operad on a vector space. Let $\mathcal{O}$ be an operad, a vector space V is said to be an $\mathcal{O}$-*algebra* if there is a morphism of operads $\mu : \mathcal{O} \rightarrow \mathcal{E}nd_V$. That is to say, there exists a map $\mu : \mathcal{O}(n) \rightarrow \mathcal{E}nd_V(n)$, one for each $n \geq 1$, such that μ preserves all of the structures.

There is a natural notion of a *V-module, M, where V is an $\mathcal{O}$-algebra*. The composition maps of a V-module are defined by taking the axioms satisfied by the composition maps of an $\mathcal{O}$-algebra and replacing the V associated to one input and the output with M, and demanding that the axioms of an algebra hold (see [10]) except that one does not allow compositions between elements in M. In other words, the axioms of a V-module M are those which make the algebra V itself naturally into a V-module.

1. Conformal Field Theory and Vertex Operator Algebras

A ($c = 0$ tree level) conformal field theory is an algebra over an operad of moduli spaces of configurations of holomorphically embedded disks in a Riemann sphere.

Let D be the (closed) unit disk in the complex plane. Let $\mathcal{P}(n)$ be the moduli space of configurations of $(n + 1)$-distinct, ordered biholomorphic embeddings of D into the Riemann sphere, $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ (in a standard coordinate z) whose images may overlap at the boundaries of the disks where any two such configurations are

identified if they are related by a automorphism of the Riemann sphere, *i.e.* a complex projective transformation, where $\mathrm{PSL}(2, \mathbb{C})$ acts upon $\mathbb{CP}^1$ by

$$z \mapsto \frac{az + b}{cz + d}, \ \forall \, a, b, c, d \in \mathbb{C} \text{ satisfying } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1.$$

$\mathcal{P} = \{\, \mathcal{P}(n) \,\}_{n \geq 1}$ forms an operad of complex manifolds where the permutation group acts by permuting the first n embeddings and the composition maps $\mathcal{P}(n) \times \mathcal{P}(n') \to \mathcal{P}(n + n' - 1)$ taking $(\Sigma, \Sigma') \mapsto \Sigma \circ_i \Sigma'$ is defined by cutting out the $(n + 1)$st disk of Σ', the ith disk of Σ and then sewing across the boundary by $x \mapsto 1/x$, x being a standard complex coordinate on D.

A *(tree level $c = 0$) conformal field theory (CFT)*, V, is a $\mathcal{P}$-algebra, *i.e.* V is a topological vector space with a smooth morphism of operads $m : \mathcal{P} \to \mathcal{E}nd_V$. An important class of CFT's are *holomorphic conformal field theories* which are CFT's such that the morphism, m, is holomorphic. Tree level means that only genus zero Riemann surfaces appear in $\mathcal{P}$. A general tree level conformal field theory is a collection of smooth maps $m : \mathcal{P} \to \mathcal{E}nd_V$ which are equivariant under the action of the permutation group and such that compositions of elements in $\mathcal{P}$ are mapped into compositions of their images in $\mathcal{E}nd_V$ up to a projective factor. In particular, V is a projective $\mathcal{P}(1)$ module and, as a consequence, is a projective representation of the Virasoro algebra with central charge c. Thus, if m is a morphism of operads then $c = 0$. However, in the general case, V can be regarded as an honest algebra over a larger operad involving determinant line bundles which fibers over $\mathcal{P}$ (see [24]). For simplicity, we shall restrict to $c = 0$ CFT's though it is a simple matter to generalize what follows to the more general case.

There is another collection of moduli spaces closely related to $\mathcal{P}$. Let $\widetilde{\mathcal{P}}(n)$ be the moduli space of configurations of $(n+1)$-distinct ordered holomorphic coordinates (no two of which have coinciding centers) on $\mathbb{CP}^1$ where any two such configurations are identified if they are related by a complex projective transformation. The collection, $\widetilde{\mathcal{P}} := \{\, \widetilde{\mathcal{P}}(n) \,\}_{n \geq 1}$, satisfies all the axioms of an operad except that one cannot always compose elements in $\widetilde{\mathcal{P}}$. An object which satisfies the axioms of an operad whenever compositions are defined is called a *partial operad*, see Huang-Lepowsky [13]. The partial operad $\widetilde{\mathcal{P}}$ contains $\mathcal{P}$ as a suboperad since holomorphically embedded unit disks may be regarded as holomorphic coordinates. Since both $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are operads of complex manifolds, the composition maps of $\widetilde{\mathcal{P}}$ are uniquely determined by analytic continuation of the composition maps of $\mathcal{P}$. The composition of an element S in $\widetilde{\mathcal{P}}(n)$ and S' in $\widetilde{\mathcal{P}}(n')$, $S \circ_i S'$, can be obtained by identifying their boundaries. Suppose there exists a number $r > 0$ such the disk of radius r about the center of the $(n'+1)$st coordinate and the disk of radius $1/r$ about the center of the ith coordinate do not contain the centers of any other coordinates. In that case, $S \circ_i S'$ is obtained but cutting out these disks and sewing. If such an r does not exist then $S \circ_i S'$ is undefined.

It is natural to define V to be an algebra over $\widetilde{\mathcal{P}}$ if there is a morphism of partial operads $\widetilde{\mathcal{P}} \to \mathcal{E}nd_V$. We will call holomorphic, $\widetilde{\mathcal{P}}$-algebras V *(c = 0) vertex operator algebras (VOA)* although as shown by Huang and Lepowsky [13], an actual VOA has a somewhat subtler structure. Because $\mathcal{P}$ is a suboperad of $\widetilde{\mathcal{P}}$, VOA's are themselves holomorphic conformal field theories and provide an important class of examples of such theories.

1. Moduli spaces $\mathcal{P}(n)$ as a quadratic operad. The operad $\mathcal{P}(n)$, $n \geq 1$, of moduli spaces of Riemann spheres with $n+1$ holomorphically embedded disks forms a quadratic operad in the sense of Ginzburg-Kapranov [10]. This means that the operad $\mathcal{P}(n)$ is the quotient $Q(K, E, R)$ of the free $K = \mathcal{P}(1)$-operad $F(E)(n)$ by an ideal generated by defining relations R which are all quadratic, i.e., lie in $F(E)(3)$. Recall that the free operad is given by $F(E)(n) = \bigoplus_{\text{binary } n\text{-trees } T} E(T)$ where $E(T) = \bigotimes_{v \in T} E(\text{In } v)$, In v denoting the set of incoming edges of a vertex v, generated by $E = \mathcal{P}(2)$.

Let us describe generators and relations of $\mathcal{P}(n)$ in more detail: we will need this in order to find the Koszul dual operad $\mathcal{P}^!(n)$ in the next section. From now on, we will work in the category of vector spaces. In particular, we consider the vector space generated by $\mathcal{P}(n)$ instead of $\mathcal{P}(n)$ itself. The ground algebra K, the space E of generators and the space R of relations will be, thus, vector spaces, whose generators we are going to describe.

The *ground algebra $K = \mathcal{P}(1)$* is the "semigroup" algebra of the distinguished Virasoro semigroup: as a vector space, it is generated by cylinders, isomorphism classes of Riemann spheres with two nonoverlapping (counted with boundary) holomorphically embedded disks. Denote the center of the first disk by 0 and the center of the second by ∞. The operad composition endows K with the structure of a semigroup: gluing the disk around ∞ on one cylinder to the disk around 0 on the second one, along the parametrizations of the disks.

The *space E of generators* is generated by pairs of pants, which are three-holed Riemann spheres. The symmetric group S_2 acts on E by interchanging the legs and K acts on E from the left by gluing a cylinder along the circle around ∞ to the waist of the pants and K^2 acts on E from the right by gluing cylinders at their disks centered at 0 to either of the legs.

Elements in the free operad generated by E, $F(E)$, have a nice geometric realization. Elements in $F(E)(n)$ can be regarded as elements in $\mathcal{P}(n)$, finite linear combinations of spheres with $n+1$ holes, together with a set of homotopy classes of curves which cuts each sphere into pairs of pants. Such a decomposition gives rise to a binary tree with a pair of pants associated to each vertex such that the root of each vertex is associated to the 3rd hole ("the waist") on the pants and the other two ("the legs") are associated to the two incoming edges.

The *space R of relations* is the subtlest. First, we need to describe the space

$F(E)(3)$, where they live. As a vector space, it is generated by the following elements. Each element is an equivalence class of a triple (T, p_1, p_2), where T is a binary 3-tree, one of the following three:
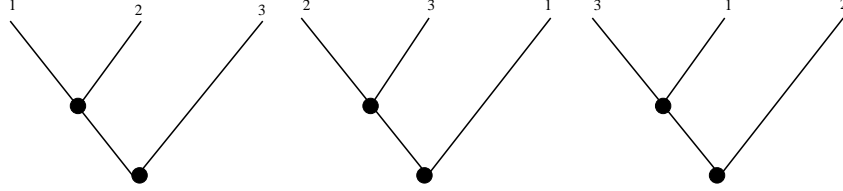

FIGURE 1. Trees Indexing $F(E)(3)$

Here p_1 and p_2 are two pairs of pants, associated to the two vertices of the tree T: p_1 to the lower vertex, p_2 to the upper one. One can think of the triple (T, p_1, p_2) as a Riemann sphere with four holomorphic disks along with a curve cutting it into two pairs of pants:
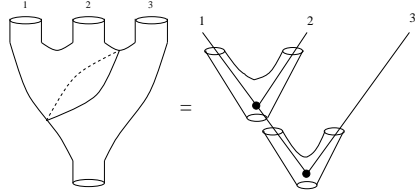

FIGURE 2. An Element of F(E)(3) and Its Associated Tree

Two triples are considered equivalent if they differ by the following diagonal action of K:

$$(T, p_1 \circ_1 k, p_2) \sim (T, p_1, k \circ p_2).$$

Geometrically, this action slides the cut on the sphere with four holes along the tube that it is wrapped around. Thus, the space $F(E)(3)$ may be described as generated by isomorphism classes of four-holed spheres with homotopy classes of cuts into two pairs of pants.

The space R of relations may be then described as follows:

$$R = \bigoplus_{\left\{\substack{\text{classes of 4-} \\ \text{holed spheres}}\right\}} \left\{ \sum_\gamma a_\gamma e_\gamma \ \middle| \ \substack{a_\gamma = 0 \text{ for all but a finite} \\ \text{number of } \gamma, \ \sum_\gamma a_\gamma = 0} \right\},$$

where the sum over γ is the sum over homotopy classes of curves of the 4-holed sphere which cut it into two pairs of pants. In fact, R is generated by two types of "4-point" relations. The first (see figure 3) corresponds to relations between different trees while the second (one of which is depicted in figure 4) corresponds to relations between the same type of tree but with different homotopy classes of curves separating the same 4-holed sphere into pairs of pants.

Theorem 1 (Coherence Theorem). *The operad $\mathcal{P}(n)$ is quadratic, with pairs of pants as generators for E and the "four-point" relations as defining relations.*

Proof. We have a natural surjective morphism of operads

$$Q(K, E, R) = F(E)/(R) \rightarrow \mathcal{P}$$

where (R) is the ideal in $F(E)$ generated by R. To show that this is an isomorphism, we have to show that all relations between elements of $F(E)(n)$ are consequences of R. Suppose we have a relation between two elements of $F(E)(n)$, corresponding to binary trees T_1 and T_2. This means we have a Riemann sphere with $n + 1$ holes and two decompositions into pairs of pants on it. Applying the four-point identities of figures 3 and 4 to fragments of the trees, we can get from T_1 to T_2. On the Riemann sphere, we can interpret those moves as erasing the cut between two subsequent pairs of pants and cutting them along another cut separating another pair of vertices on the same four-point fragment. Finally, we arrive at two decompositions into pairs of pants of the same $n + 1$-holed Riemann sphere corresponding to one and the same tree.

Now, working from the root to the top, we can make the cuts homotopic if we use the relations 4 corresponding to one and the same tree and different choices of homotopy classes of cuts separating holes a and b. These relations follow from 4.

Thus, we have two decompositions into pairs of pants where cuts are all homotopic. Finally we can use elements of K to move the cuts along the legs to get identical decompositions. This proves the theorem. $\square$

2. The Koszul dual $\mathcal{P}^!(n)$. The Koszul dual to a quadratic operad $Q(K, E, R)$ is the quadratic operad $Q(K^{\mathrm{op}}, E^\vee, R^\perp)$, cf. Ginzburg-Kapranov [10]. Here K^{op} is the algebra K with the reversed multiplication. Geometrically, we can regard this as PK, where P interchanges the input and output of a cylinder.

Here is how the Koszul dual is defined. For a (K, K^n)-module V with an S_n action, we consider the $(K^{\mathrm{op}}, (K^{\mathrm{op}})^n)$-module $V^\vee = \mathrm{Hom}_K(V, K)$, K in the subscript acting on K by the left multiplication and on V via the left action. We will provide $V^\vee$ with the natural structure of an S_n-module, twisted by the sign representation.

Having our (K, K^2)-module E with an S_2 action, we take the space $E^\vee$ as the generating space of the Koszul dual. Using the natural mapping $F(E^\vee)(3) \rightarrow F(E)(3)^\vee$, we can define $R^\perp \subset F(E^\vee)(3)$ as the annihilator of $R \subset F(E)(3)$.

Proposition 2.

$$R^\perp = \mathcal{P}(3)^\vee,$$

that is, $R^\perp$ consists of K-valued K-equivariant functions $C(S)$ of classes S of 4-holed spheres.

Thus, the space $\mathcal{P}^!(n)$ is a quotient of the space of K-valued K-equivariant functionals $C(S, \gamma)$ of an isomorphism class S of a Riemann sphere with $n+1$ holes and a homotopy class γ of a decomposition of S into pairs of pants. The space of relations of this quotient space is the ideal generated by $\mathcal{P}(3)^\vee$.

Proposition 3. *For all $n \neq 1$, $\mathcal{P}^!(n)$ is the quotient space of K-valued K-equivariant functions $f(S, \gamma)$, $S \in \mathcal{P}(n)$, γ being a partition of S into pairs of pants corresponding to a tree diagram, considered up to homotopy, by the subspace generated by functions $g_T(S, \tilde{\gamma})$, where $S \in \mathcal{P}(n)$ and $\tilde{\gamma}$ is a partition of S into one 4-holed sphere and a few pairs of pants corresponding to a tree T with a marked 3-subtree. In particular, $\mathcal{P}^!(1) = \mathcal{P}(1)^{\mathrm{op}}$, $\mathcal{P}^!(2) = \mathcal{P}(2)^\vee$ and $\mathcal{P}^!(3) = F(\mathcal{P}(2))(3)^\vee / \mathcal{P}(3)^\vee$.*

3. The operad cohomology. For a quadratic operad $\mathcal{O} = Q(K, E, R)$ and an $\mathcal{O}$-algebra A and a module M over A, define a cochain complex with the nth term

(1) $\quad C^n = C_{\mathcal{O}}^n(A, M)$
$$= \mathrm{Hom}_K(F(E^\vee)(n)^\vee \otimes_{S_n, K^n} A^{\otimes n}, M)/(\langle R \rangle(n) \otimes_{S_n, K^n} A^{\otimes n})^\perp, \qquad n \geq 1,$$

where, by a certain abuse of notation, $F(E^\vee)(n)^\vee := \bigoplus_{\text{binary } n\text{-trees } T} E(T) \otimes \det(T)$, and $\det(T)$ is the top exterior power of the vector space generated by all of the internal edges of a tree T (those edges which are not inputs to T). Also, K here acts by the left action on $F(E^\vee)(n)^\vee$ and M. The subspace $\langle R \rangle(n) \subset F(E^\vee)(n)^\vee$ is defined as follows:

$$\langle R \rangle(n) = \bigcap_{\text{1-ternary } n\text{-trees } T} (R \otimes \det)(T),$$

where a 1-ternary tree T is a tree with 1 ternary vertex u and binary vertices v otherwise and

$$(R \otimes \det)(T) = (R \otimes \det)(\mathrm{In}\, u) \otimes \bigotimes_{\text{binary } v \in T} E(\mathrm{In}\, v) \otimes \det(\mathrm{In}\, v) \otimes \det\{\text{terminal edges}\},$$

$(R \otimes \det)(\mathrm{In}\, u)$ being the same as R except that it is labeled by the incoming edges $\mathrm{In}\, u$ of the vertex u and that it is twisted by the det.

Remark 1. If K were a field and the space E finite dimensional over K, the space $\langle R \rangle$ would be the same as $(R^\perp)^\perp$, the subspace in $F(E^\vee)^\vee$ which vanishes when contracted with elements in the operad ideal $(R^\perp) \subset F(E^\vee)$. The following dual description of the cochain complex would also be true:

$$C^n \cong \mathrm{Hom}_K(\langle R \rangle(n) \otimes_{S_n, K^n} A^{\otimes n}, M)$$
$$\cong \mathrm{Hom}_{S_n, K^n}(A^{\otimes n}, \mathrm{Sgn}_n \otimes \mathcal{O}^!(n) \otimes_K M),$$

Sgn_n being the sign representation of S_n.

The differential $d : C^{n-1} \to C^n$ can be defined as follows. A vertex v of a tree T is called *extremal*, if all of its incoming edges are terminal for the whole tree. For an extremal vertex v of a tree T, let T_v denote the subtree formed by the vertex v along with its incoming edges, and T/v the tree obtained by removing T_v from T. Let $[n]$ denote the set $\{1, \dots, n\}$. We label the tree T/v by the set $[n]/v$ which consists of the elements of $[n]$ which label T, adding the label v, and omitting the labels we have just erased. Since we ultimately will consider maps which are invariant under the symmetric group, it will not matter which $n-1$-element set we used to label T/v. If there is an external edge coming into the root of the tree, we define a tree T/r by erasing the tree T_r formed by the root vertex and its incoming edges. We label the T/r similarly to T/v.

The space $F(E^\vee)(n)^\vee$ is the direct sum of the spaces $E(T) \otimes \det(T)$ of binary n-trees decorated with elements of E at vertices. The action of $\mathcal{O}$ on A defines

$$E(T_v) \otimes A \otimes A \to A$$

and thereby

$$d_v : \mathrm{Hom}(E(T/v) \otimes \det(T/v) \otimes A^{\otimes[n]/v}, M) \to \mathrm{Hom}(E(T) \otimes \det(T) \otimes A^{\otimes n}, M),$$

where for the determinants we used the mapping

$$\det(T) \to \det(T/v)$$
$$e_1 \wedge \cdots \wedge e_{n-2} \mapsto e_1 \wedge \cdots \wedge \hat{e}_k \wedge \dots e_{n-2},$$

e_k being the internal edge of T missing from T/v.

Similarly, for each external edge e coming into the root, we have our module structure morphism

$$E(T_r) \otimes A \otimes M \to M$$

and

$$d_r^e : \mathrm{Hom}(E(T/r) \otimes \det(T/r) \otimes A^{\otimes[n]/r}, M) \to \mathrm{Hom}(E(T) \otimes \det(T) \otimes A^{\otimes n}, M).$$

If there are no external edges coming into the root of T, we put $d_r^e = 0$. For a binary tree T, two external edges may come into the root, if and only if T is a 2-tree. This is the only case when there is more than one mapping d_r^e.

Thus we can form a mapping $d = \sum_v d_v + \sum_e d_r^e$

$$d : \mathrm{Hom}_K(F(E^\vee)(n-1)^\vee \otimes_{S_{n-1},K^{n-1}} A^{\otimes n-1}, M) \to \mathrm{Hom}_K(F(E^\vee)(n)^\vee \otimes_{S_n,K^n} A^{\otimes n}, M).$$

The differential $d : C^1 = \mathrm{Hom}_K(A, M) \to C^2 = \mathrm{Hom}_K(E \otimes_{S_2,K} A \otimes A, M)$ may need special attention, as it involves cutting cylinders off a pair of pants. If $c : A \to M$ is a 1-cochain, then

$$(dc)(S; a_1, a_2) = X(S; a_1, c(a_2)) - c(Y(S; a_1, a_2)) + X(\tau S; a_2, c(a_1)),$$

where $S \in E$, a_1, $a_2 \in A$, $X : E \otimes A \otimes M \to M$ is the module structure morphism, $Y : E \otimes A \otimes A \to A$ is the algebra structure morphism, and $\tau \in S_2$ is the transposition.

Lemma 4. *The subspaces*

$$(\langle R \rangle(n) \otimes_{K^n} A^{\otimes n})^{\perp}$$

are preserved by d and the resulting map on the quotients C^n is a differential, i.e., $d^2 = 0$.

Proof. If we have a cochain c in C^n which vanishes on $(R \otimes \det)(T)$ for a 1-ternary tree T, terms of its differential dc will vanish on $(R \otimes \det)(T')$ for 1-ternary $n+1$-trees T' obtained from T by grafting binary trees onto it. Therefore, they will vanish on $\langle R \rangle(n + 1)$ as well.

For a cochain $c \in C^n$, those terms of $d^2 c$ which are obtained by erasing the vertices of a 1-ternary tree T whose ternary vertex is terminal will vanish on $(R \otimes \det)(T)$, because R as the relation spaces acts on A^3 by zero. There are also terms in $d^2 c$ corresponding to the root vertex - they will vanish on $(R \otimes \det)(T)$ corresponding to a 1-ternary tree T whose ternary vertex is the root vertex, because R acts by zero on $A^2 \otimes M$. Terms of $d^2 c$ whose terms are too far away to be united in a 3-subtree, will cancel each other because of the signs imposed by $\det(T)$. $\square$

Definition 1. The *operad cohomology* $H^n_{\mathcal{O}}(A, M)$, $n \geq 1$, of an algebra A over an operad $\mathcal{O}$ with values in a module M is the cohomology of the complex $C^n_{\mathcal{O}}(A, M)$.

Remark 2. When $\mathcal{O} = \mathcal{A}s$, we obtain the Hochschild complex

$$\mathrm{Hom}(A, M) \to \mathrm{Hom}(A^2, M) \to \dots,$$

when $\mathcal{O} = \mathcal{L}ie$, we obtain the standard complex

$$\mathrm{Hom}(A, M) \to \mathrm{Hom}(\Lambda^2 A, M) \to \dots,$$

when $\mathcal{O} = \mathcal{C}omm$, we obtain the Harrison complex

$$\mathrm{Hom}(A, M) \to \mathrm{Hom}(S^2 A, M) \to \dots.$$

Thus the operad cohomology in these cases is Hochschild, Lie algebra and Harrison, respectively. We expect any other classical and new examples to fit this scheme. For instance, applying the above to the noncommutative Poisson operad $\mathcal{O} = \mathcal{P}oiss$, we get the complex giving the cohomology of a noncommutative Poisson algebra, see [6]. These complexes are missing their zeroth terms, which will be added in the following section.

4. Digression: the zeroth term. This section is a brief discussion how to modify the formalism of quadratic operads, so as to include the zeroth term in the cochain complex. It is in fact digression from our main operad $\mathcal{P}$ of moduli spaces, because it is not symmetric in the following sense.

A *symmetric quadratic operad with vacuum* $\mathcal{O} = Q(C, K, E, R)$ is a quadratic operad with $\mathcal{O}(1) = K$, the space E of generators and the space $R \subset F(E)(3)$ of defining relations to which a *vacuum space* $\mathcal{O}(0) = C$ is added. The composition maps are extended so that they satisfy the associativity axioms and the equivariance under the action of the permutation groups. The word symmetric menas that in addition, *the two right actions* $\circ_1$ *and* $\circ_2$ *of* K *on* E *must coincide.*

We also need the following data and axioms, satisfied in all examples of Remark 2, see the examples below.

> A morphism $\phi = \circ_1 : E \otimes_K C \to K$ of (K, K)-modules, K acting on E via the right module structure, along with a right inverse $s : K \to E \otimes_K C$, $\phi s = \mathrm{id}_K$.

Here is the axiom relating the above data associated with C to R. *We require that the image of the composition of certain maps*

(2) $$E \xrightarrow{\delta_1} (E \otimes_K K)^{\oplus 3} \xrightarrow{\delta_0} F(E)(3) \otimes_K C$$

be contained in $R \otimes_K C$. Here we should think of the three components of the middle term as being indexed by the three trees shown in figure 5, all of which are allowed to have vertices of valence 1.

The mappings δ_1 and δ_0 are defined as follows:

$$\delta_1(e) = (e \otimes \mathrm{id})_1 - (e \otimes \mathrm{id})_2 + (e \otimes \mathrm{id})_3,$$

the outer indices refer to the component of $(E \otimes_K K)^{\oplus 3}$ (see figure 6).

The action of δ_0 on a component of $(E \otimes_K K)^{\oplus 3}$ corresponding to a tree T is equal to

$$\delta_0 = \mathrm{id}_E \otimes s - \tau \otimes s$$

where s acts by replacing the 1-corolla (the subtree consisting of a vertex with exactly one incoming edge) by a difference of two trees as shown in figure 7 and τ is the transposition in S_2 acting on E. The assumption that the two right actions of K on E are equal makes the second term of δ_0 correctly defined.

In particular, δ_0 acts on the first component of $(E \otimes_K K)^{\oplus 3}$ as shown in figure 8 and similarly for the other two components.

An algebra over such an operad $\mathcal{O}(n)$ is defined as a vector space V along with a morphism $\{\mathcal{O}(n) \to \mathcal{E}nd_V(n) \mid n \geq 0\}$ respecting all of the data.

Let us illustrate the above abstract nonsense with the examples of the standard algebraic operads $\mathcal{A}s$, $\mathcal{L}ie$ and $\mathcal{C}omm$, describing associative, Lie and commutative algebras.

Example 1 (The associative operad $\mathcal{A}s$). Consider the alphabet $\{x_1, x_2, \dots\}$ and words constructed from it. Then C is generated by the empty word $()$ and K by the word x_1 - both spaces are one-dimensional. E is generated by two words x_1x_2 and x_2x_1, $F(E)(3)$ is generated by all possible permutations of the words $(x_1x_2)x_3$ and $x_1(x_2x_3)$ and R is generated by 6 associators of the kind $(x_1x_2)x_3 - x_1(x_2x_3)$. The mappings ψ and $\eta_{1,2,3}$ are obvious, because we have the identity element $\mathrm{id} = x_1 \in K$, the mapping ϕ takes both $x_1x_2 \otimes ()$ and $x_2x_1 \otimes ()$ to x_1, and $s : x_1 \mapsto x_1x_2 \otimes ()$. The interested reader may check that $\delta_0\delta_1$ maps x_1x_2 to the linear combination of three associators: $(x_1x_3)x_2 - x_1(x_3x_2) - ((x_3x_1)x_2 - x_3(x_1x_2)) - ((x_1x_2)x_3 - x_1(x_2x_3)) \in R$, tensored with $()$.

Example 2 (The Lie operad $\mathcal{L}ie$). In this case C is generated by $[\,]$, K by x_1, E by $[x_1, x_2]$ and R by the Jacobi identity, see [10] for more details. Since C, K and E are naturally identified with complex numbers $\mathbb{C}$, we can define all mappings ϕ, s, ψ and η's as $\mathrm{id} : \mathbb{C} \to \mathbb{C}$. Then $\delta_0\delta_1$ maps $[x_1x_2]$ to twice the Jacobi identity, tensored with $[\,]$.

Example 3 (The commutative operad $\mathcal{C}omm$). This case is similar to $\mathcal{L}ie$, but here δ_0 is already zero, because $x_1x_2 = x_2x_1$.

Now we are up to add the zeroth term to the complex $C^\bullet = C^\bullet_{\mathcal{O}}(A, M)$. By definition,

$$C^0 = C^0_{\mathcal{O}}(A, M) = \mathrm{Hom}_K(C, M).$$

The differential

$$d : C^0 = \mathrm{Hom}_K(C, M) \to C^1 = \mathrm{Hom}_K(A, M),$$

is defined similar to all higher differentials above. There are no terms d_v corresponding to extremal vertices, but there are two terms d^1_r and d^2_r corresponding to the root. The term d^1_r is the composition of

$$\mathrm{Hom}(C, M) \to \mathrm{Hom}(E \otimes_K C \otimes_K A, M),$$

which is induced by the structure morphism

(3) $$E \otimes A \otimes M \to M,$$

and

$$\mathrm{Hom}((E \otimes_K C) \otimes_K A, M) \to \mathrm{Hom}(K \otimes_K A, M),$$

which is induced by the structure morphism

(4) $$s : K \to E \otimes_K C.$$

The term d^2_r is merely $(d^1_r)^\tau$, where we compose s with the transposition $\tau \in S_2$ acting on E.

The axiom (2) ensures that $d^2 = 0$, as in Lemma 4. Thus we now have a complex

$$0 \to C^0_{\mathcal{O}}(A, M) \to C^1_{\mathcal{O}}(A, M) \to C^2_{\mathcal{O}}(A, M) \to \ldots,$$

whose cohomology groups $H^n_{\mathcal{O}}(A, M)$, $n \geq 0$, form the *cohomology of an algebra over a symmetric operad with vacuum.*

Remark 3. In the above cases of operads $\mathcal{O} = \mathcal{A}s$, $\mathcal{L}ie$ and $\mathcal{C}omm$, adding the zeroth term to the complex $C^\bullet_{\mathcal{O}}(A, M)$ completes the complexes of Remark 2 by including the standard zeroth terms. For the Harrison complex, the operator $d : C^0 \to C^1$ will be equal to zero, as usual.

5. Description of the complex for the moduli space operad. Here we are going to apply the construction of the previous sections to the operad $\mathcal{P}(n)$, $n \geq 1$, of isomorphism classes of Riemann spheres with $n+1$ labeled holomorphically embedded disks.

Suppose we have a (tree level $c = 0$) conformal field theory (CFT), which is nothing but an algebra A over the operad $\mathcal{P}(n)$, and a module M over this algebra. One may require these algebras and modules be smooth, i.e., the action of the operad $\mathcal{P}(n)$ depends smoothly on the point in $\mathcal{P}(n)$, which may be easily included in our formalism. We do not require smoothness for two reasons: the noncontinuous cohomology of Lie groups makes perfect sense and the action of the Virasoro algebra, which is assumed implicitly through the action of $K = \mathcal{P}(1)$, may be regarded as a kind of smoothness condition - $\mathcal{P}(n)$ is a locally homogeneous space over $n+1$ copies of the Virasoro algebra, which acts through K^{n+1}, and the equivariance of algebra and module structure maps as well as cochains with respect to K^{n+1} will give rise to a stronger condition than just smoothness.

We may also consider holomorphic algebras and modules over $\mathcal{P}(n)$ and holomorphic cochains, which brings us to the case of chiral or vertex operator algebras and modules over them. This is one of the main points of the second part of the paper.

We will consider cochains (1), and our description of cochains will be similar to that of the Koszul dual operad $\mathcal{P}^!(n)$ in Section 2.

The space $C^n_{\mathcal{P}}(A, M)$ of cochains of a CFT, $n \geq 1$, is a quotient space of the space $\mathrm{Hom}_K(F(E^\vee)(n)^\vee \otimes_{S_n, K^n} A^{\otimes n}, M)$ of K-equivariant functions

$$(5) \qquad\qquad f(S, \gamma, e_T; a_1, a_2, \ldots, a_n)$$

where $S \in \mathcal{P}(n)$, γ is a partition of S into pairs of pants corresponding to a tree diagram T, considered up to homotopy, $e_T = e_1 \wedge \cdots \wedge e_{n-2} \in \det(T)$, $e_1, \ldots, e_{n-2}$ being internal edges of T, and $a_1, \ldots, a_n \in A$. Including e_T as a variable may be interpreted as choosing an orientation on the space generated by internal edges. The functions $f(\sigma S, \gamma, e_T; a_1, a_2, \ldots, a_n)$ and $f(S, \gamma, e_T; a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)})$, where σ is a

permutation acting on S by relabeling the inputs, are said to be equivalent. Functions of the form

$$f((S, \gamma, e_T) \circ_i x; a_1, \ldots, a_i, \ldots, a_n) \text{ and } f(S, \gamma, e_T; a_1, \ldots, xa_i, \ldots, a_n)$$

for $x \in K$ are also regarded as equivalent. The subspace by which we mod out is the sum of the following subspaces each of which is indexed by a 1-ternary n-tree $\tilde{T}$, i.e., a tree whose vertices are all binary with the exception of one which is ternary.

The subspace corresponding to a 1-ternary tree $\tilde{T}$ is the space of functions

$$g(S, \tilde{\gamma}, e_{\tilde{T}}; a_1, a_2, \ldots, a_n) \in M,$$

where $\tilde{\gamma}$ is a partition of S into one 4-holed sphere and some number of pairs of pants corresponding to the tree $\tilde{T}$. Such functions form a subspace of $\operatorname{Hom}_K(F(E^\vee)(n)^\vee \otimes_{S_n, K^n} A^{\otimes n}, M)$, because if γ is a partition of S into pairs of pants, which coincides with $\tilde{\gamma}$ after erasing one cut, then we set $g(S, \gamma, e_T; \ldots) = g(S, \tilde{\gamma}, e_{\tilde{T}}; \ldots)$, where $e_T = e_{\tilde{T}} \wedge e_s$, s being the edge of T missing in $\tilde{T}$. We set $g(S, \gamma, e_T; \ldots) = 0$ otherwise.

The terms C^1 and C^2 consist of functions

$$f(S; a_1, \ldots, a_n) \in M,$$

where S is a cylinder when $n = 1$ and a pair of pants when $n = 2$. For $n = 3$, C^3 is the quotient of the space of functions (5) by the subspace of functions which are independent of the partition into pairs of pants.

The differential acts as follows. Denote by $Y(S; a_1, a_2)$ the $\mathcal{P}$-algebra structure

$$Y : E \otimes A \otimes A \to A$$

on A and by $X(S, a, m)$ the A-module structure

$$X : E \otimes A \otimes M \to M$$

on M. For an n-cochain f, as in (5), $n \geq 2$,

$$(df)(S, \gamma, e_T; a_1, a_2, \ldots, a_{n+1})$$
$$= \sum_{\text{extremal vertices } v} f(S_1^v, \gamma^v, e_{T/v}; Y(S_2^v; a_i, a_j), a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_{n+1})$$
$$+ \sum_{\left\{\begin{array}{l}\text{terminal edges}\\ e \text{ pointing to}\\ \text{the root}\end{array}\right\}} X(S_1^e; a_i, f(S_2^e, \gamma^e, e_{T/r}; a_1, \ldots, \hat{a}_i, \ldots.a_{n+1})),$$

where (S, γ) is an $n+2$-holed Riemann sphere partitioned into pairs of pants, the first summation runs over extremal vertices v of the diagram (S, γ), i and j are the labels of the vertices right above v, S_2^v is the pair of pants corresponding to v, (S_1^v, γ^v) is what remains of (S, γ) after removing S_2^v, and T/v is the tree corresponding to (S_1^v, γ^v). The holes of S_1^v are labeled as follows. The hole left after removing S_2^v goes

first, then all others following their order on S. $e_{T/v}$ is just the wedge product of the internal edges of T/v in the same order they were in e_T. The holes of S_2^v labeled with i on S is labeled by 1, the other input hole by 2. Similarly, the second summation runs over terminal edges e going into the root of (S, γ), if any, i is the label at the input of the edge e, S_1^e is the pair of pants corresponding to the root and (S_2^e, γ^e) is what remains after removing S_1^e from (S, γ). Also, T/r is the tree corresponding to (S_2^e, γ^e), the holes of S_2^e are labeled in the same order they were on S, the hole labeled with i on S is labeled with 1 on S_1^e, and $e_{T/r}$ is the product of the corresponding edges of T in the same the order.

The differential for 1-cochains is defined as at the end of Section 3. Lemma 4 guarantees that the differential d is well-defined and that $d^2 = 0$. This explicit description wil be helpful when we study deformations of CFT's.

6. The cohomology of TFT's and the Harrison cohomology. A (tree level) *topological field theory* (TFT) is a $\mathcal{P}$-algebra A, such that the structure morphisms $\mathcal{P}(n) \to \mathcal{E}nd_A$ are constant. This implies that the Virasoro semigroup $\mathcal{P}(1)$ acts trivially on A and that the action of a pair of pants from $\mathcal{P}(2)$ defines the structure of a commutative algebra on A. In this particular case, the above complex $C_{\mathcal{P}}^{\bullet}(A, M)$ turns into the Harrison complex $C_{\mathrm{H}arr}^{\bullet}(A, M)$ of a commutative algebra A with values in a module M. The second cohomology group $\mathrm{Harr}^2(A, A)$ parametrizes classes of infinitesimal deformations of A within the category of commutative algebras. From Theorem 5 below, it will follow that the same group parameterizes deformations of a TFT, as well.

7. Deformations of conformal field theories. We will now show that the complex $C_{\mathcal{P}}(A, A)$ governs the deformations of $\mathcal{P}$-algebras. Let A be a $\mathcal{P}$-algebra where $\mathcal{P} = \{ \mathcal{P}(n) \}_{n \geq 1}$ and the morphism is denoted by $Y : \mathcal{P} \to \mathcal{E}nd_A$. Since $\mathcal{P}$ is a quadratic operad, the algebra structure is completely determined by K-module structure of A and the action of E on A. Let A be a $\mathcal{P}$-algebra then A is a K-module. The problem is to classify inequivalent infinitesimal deformations of A as a $\mathcal{P}$-algebra keeping the K-module structure fixed. An *infinitesimal deformation of a CFT* A is the structure of a $\mathcal{P}$-algebra on $A \otimes \mathbb{C}[t]/(t^2)$ over the algebra $\mathbb{C}[t]/(t^2)$ of *dual numbers*, such that the reduction of this structure at $t = 0$ coincides with the initial $\mathcal{P}$-algebra structure on A over $\mathbb{C}$. Two infinitesimal deformations are said to be *equivalent*, if there is an isomorphism between the two $\mathcal{P}$-algebras over $\mathbb{C}[t]/(t^2)$ inducing the identity morphism at $t = 0$. *Keeping the K-module structure on A fixed* means that we assume this structure is extended trivially to $A \otimes \mathbb{C}[t]/(t^2)$.

Infinitesimally, a K-module structure on A is the structure of a module over the Virasoro algebra $\mathrm{Lie}(K)$. Classes of infinitesimal deformations of the K-module structure on A are parameterized by the Lie algebra cohomology $H^1(\mathrm{Lie}(K); \mathrm{End}(A))$. Writing down conditions for a deformation of a CFT, where the action of K is allowed to vary, is an easy exercise. However, its cohomological interpretation is not

obvious and must combine the operad cohomology and the Lie algebra cohomology in a new kind of spectral sequence.

It is worth pointing out that for the arguments in this section, it makes no difference whether $\mathcal{P}$ is the moduli space of Riemann spheres with holomorphically embedded disks which are allowed to intersect at their boundaries or whether $\mathcal{P}$ is the same but where the disks are not allowed to intersect at all.

The relations R give rise to an associativity condition on the binary operations on A. Let S be a 4-holed sphere in $\mathcal{P}(3)$ with any two partitions of S, γ and γ', into two pairs of pants in $\mathcal{P}(2)$ then the associativity relations are

$$(Y \circ Y)_{(S,\gamma)} = (Y \circ Y)_{(S,\gamma')}$$

where $Y : \mathcal{P}(2) \to \mathcal{E}nd_A(2)$ is the algebra map and $(Y \circ Y)_{(S,\gamma)}$ denotes the composition indicated by the tree associated to (S, γ). Let $Y' = Y + t\alpha$ be a deformation of Y where α is an S_2-equivariant and (K, K^2)-equivariant (smooth) map $\mathcal{P}(2) \to \mathcal{E}nd_A(2)$. Notice that α can be interpreted as an element in $C^2_{\mathcal{P}} = \operatorname{Hom}_K(P(2) \otimes_{S_2,K^2} A^2, A)$. Keeping terms of first order in t, we obtain

$$(Y \circ \alpha + \alpha \circ Y)_{(S,\gamma)} = (Y \circ \alpha + \alpha \circ Y)_{(S,\gamma')}$$

which we can rewrite as

$$(Y \circ \alpha + \alpha \circ Y)_{(S,\gamma-\gamma')} = 0.$$

This means that

$$(d\alpha)_r := (Y \circ \alpha + \alpha \circ Y)_r = 0$$

for any r in R. Interpreting $d\alpha$ as an element of $\operatorname{Hom}_K(F(E)(3) \otimes_{S_3,K^3} A^{\otimes 3}, A)$ the last equation means that $d\alpha$ belongs to $(R \otimes_{K^3} A^{\otimes 3})^{\perp}$ and, therefore, vanishes in $C^3_{\mathcal{P}}(A, A)$. In other words, α is a cocycle in $C^2_{\mathcal{P}}(A, A)$.

However, it may be that the deformation given above is trivial. This means that there exists an isomorphism $\Phi = \operatorname{id}_A + t\beta : A \to A$, $\beta \in \operatorname{Hom}_K(A, A) = C^1_{\mathcal{P}}(A, A)$, satisfying

$$\Phi \circ Y' = Y \circ (\Phi \otimes \Phi).$$

Plugging in $Y' = Y + t\alpha$ and we obtain for all S in E and v_1, v_2 in A

$$\alpha(S; v_1, v_2) = Y(S; v_1, \beta(v_2)) - \beta(Y(S; v_1, v_2)) + Y(S; \beta(v_1), v_2)$$

but the right hand side is nothing more than $d\beta$. We have just shown the following.

Theorem 5. *The equivalence classes of infinitesimal deformations of a (tree level $c = 0$) conformal field theory (or $\mathcal{P}$-algebra) A which leave the action of the Virasoro semigroup fixed are classified by elements in $H^2_{\mathcal{P}}(A, A)$.*

Remark 4. In accordance with the general philosophy behind deformation theory, cf. Deligne [1, 2], Drinfeld [5], Goldman-Millson [11], Kontsevich [17] and Schlessinger-Stasheff [23], the complex $C_{\mathcal{P}}^{\bullet}(A, A)$, governing deformations of a $\mathcal{P}$-algebra A has a natural structure of a differential graded Lie algebra. This is because $C_{\mathcal{P}}^{\bullet}(A, A)$ is obviously an operad, and the compositions $\circ_i$ on it define a Lie bracket by the formula:

$$[f, g] = f \circ g \pm g \circ f,$$
$$f \circ g = \sum_i \pm f \circ_i g.$$

The signs are almost impossible to figure out, as usual. It is better to refer to [20, 7], where this kind of structure for an operad has been observed. Moreover, results of [7] imply a richer structure of a brace algebra on $C_{\mathcal{P}}^{\bullet}(A, A)$. Another way to see this bracket, as suggested to us by Stasheff, is to generalize the results of his paper [25] to the operadic setting. This means to construct a canonical isomorphism $C_{\mathcal{P}}^{\bullet}(A, A) \to \mathrm{Coder}(BA)$, where BA is the bar construction for the $\mathcal{P}$-algebra A, and to carry the commutator of coderivations over to the cochain complex.

2. Deformations and the Cohomology of Algebras over the Little Intervals Operad and Vertex Operator Algebras

Having shown that the cohomology theory associated to $\mathcal{P}$-algebras governs deformations of $\mathcal{P}$-algebras, we shall now do the reverse – we shall obtain the cohomology of algebras over the little intervals operad by studying its deformations mimicking the construction of Hochschild cohomology from deformations of associative algebras. We begin by realizing the operad governing associative algebras as a certain moduli space of configurations of embedded intervals in the oriented circle and then realize the little intervals operad as a suboperad of a real analog of the operad governing conformal field theory. In the context of vertex operator algebras, this leads us to a complex which closely resembles Hochschild cohomology.

For simplicity, the operads we will consider henceforth will be operads with unit of the form $\mathcal{O} = \{ \mathcal{O}(n) \}_{n \geq 1}$ *although many of these arguments extend to the general case.*

1. The associative operad as a one dimensional topological field theory. *Henceforth, let $\mathbb{R}$ be the real line oriented so that the canonical inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$ is orientation preserving and let I be the oriented interval $[-1, 1]$. Consider the oriented circle S^1. For all $n \geq 1$, let $\mathcal{A}(n)$ be the space of configurations of $(n + 1)$ distinct, ordered, orientation preserving embeddings of I into S^1 where any two such configurations are identified if they are related by an orientation preserving diffeomorphism of S^1. S_n acts upon $\mathcal{A}(n)$ simply transitively by permuting the ordering of the first n maps. There are composition maps $\mathcal{A}(n) \times \mathcal{A}(n') \to \mathcal{A}(n + n' - 1)$*

taking $(\Sigma, \Sigma') \mapsto \Sigma \circ_i \Sigma'$ given by cutting out the $(n' + 1)$st interval of Σ' and the ith interval of Σ and then gluing them together along their boundaries in an orientation preserving way. These operations make $\mathcal{A} = \{\,\mathcal{A}(n)\,\}_{n\geq 1}$ into an operad. The orientation on S^1 picks out a canonical element Θ_n in $\mathcal{A}(n)$ for each n which is the unique class consisting of maps whose ordering increases as one proceeds around S^1 as prescribed by its orientation. (See figure 9.) This element allows us to identify $\mathcal{A}(n)$ with S_n

An $\mathcal{A}$-algebra is nothing more than an associative algebra, hence, $\mathcal{A}$ may be called the *associative operad*. $\mathcal{A}$ generates an operad of vector spaces $\mathcal{A}s = \{\,\mathcal{A}s(n)\,\}_{n\geq 1}$ which is the operad that is usually referred to as the associative operad, in the literature.

$\mathcal{A}s$ is a quadratic operad generated by $(\mathcal{A}s(1), \mathcal{A}s(2))$ with relations

$$(6) \qquad\qquad \Theta_3 \;=\; \Theta_2 \circ_1 \Theta_2 = \Theta_2 \circ_2 \Theta_2$$

as well as similar relations which arise from the action of S_3 on this equation (see figure 10). This means that all of the operations of an $\mathcal{A}s$-algebra can be obtained from the identity map, compositions of binary multiplications and the action of the permutation group.

2. Deformations of associative algebras and Hochschild cohomology.
In this section, we shall review the manner in which an infinitesimal deformation of an associative algebra gives rise to Hochschild cohomology.

Let V be an $\mathcal{A}$-algebra with a morphism of operads $\mu : \mathcal{A} \to \mathcal{E}nd_V$. Let $\rho := \mu_{\Theta_1} = \mathrm{id}_V$, identity map on V, and let the linear map $m := \mu_{\Theta_2} : V^{\otimes 2} \to V$ let be the *multiplication operation*. Since $\mathcal{A}$ is a quadratic operad, μ is completely determined by compositions of ρ and m. The associativity relations (equation 6) imply that V is an associative algebra, *i.e.* the multiplication satisfies

$$(7) \qquad\qquad m \circ_1 m = m \circ_2 m.$$

We shall now deform the associative algebra V infinitesimally. Let $\mu \mapsto \mu' = \mu + t\alpha$ where t is a small parameter and α an element in $\mathcal{E}nd_V(2)$ then impose that μ' make V into an associative algebra up to first order in t. Since μ' is entirely determined by $m' := \mu'_{\Theta_2}$ (and the unit element which remains fixed), we need only impose associativity of m' (equation 7, replacing m by m') keeping terms of first order in t. Putting all terms on one side of the equality, we obtain

$$(8) \qquad\qquad m \circ_2 \alpha - \alpha \circ_1 m + \alpha \circ_2 m - m \circ_1 \alpha = 0.$$

However, it is possible that the infinitesimal deformation α, up to first order in t, gives rise to an algebra which is isomorphic to the original, *i.e.* that the deformation

is trivial. This occurs if there exists a β in $\mathcal{E}nd_V(1)$ such that the map $\Phi := \mathrm{id}_V - t\beta$ satisfies

$$(9) \qquad\qquad m' \circ (\Phi \otimes \Phi) = \Phi \circ m.$$

Plugging in $m' = m + t\alpha$ and $\Phi = \mathrm{id}_V - t\beta +$ and keeping terms up to first order in t, we obtain

$$(10) \qquad\qquad \alpha = m \circ_2 \beta - \beta \circ_1 m + m \circ_1 \beta$$

Therefore, (nontrivial) infinitesimal deformations of V are classified by α in $\mathcal{E}nd_V(2)$ satisfying equation 8 quotiented by those which satisfy 10 for some β in $\mathcal{E}nd_V(1)$. Equation 8 is precisely the condition that α be a 2-cocycle in Hochschild cohomology of V with values in itself while equation 10 is precisely the condition that α be a 2-coboundary in V with values in itself. Therefore, the space of nontrivial infinitesimal deformations of V is determined by dimension two cohomology classes of V with values in itself. Noting that equations 8 and 10 make sense even if α were an element of $\mathrm{Hom}(V^{\otimes 2}, M)$ where M is a V-module, we are led to the following definition of the Hochschild cohomology of the associative algebra V with values in a V-module M. The space of Hochschild n-cochains is $C_{\mathcal{A}}^n(V, M) := \mathrm{Hom}(V^{\otimes n}, M)$ and the differential $d : C_{\mathcal{A}}^n(V, M) \to C_{\mathcal{A}}^{n+1}(V, M)$, whose actions on 1 and 2 cochains are given by the right hand sides of equations 8 and 10, is defined by

$$(11) \qquad\qquad d\alpha := m \circ_2 \alpha + \sum_{i=1}^{n} (-1)^i \, \alpha \circ_i m + (-1)^{n+1} m \circ_1 \alpha.$$

It is a simple exercise to verify that $d^2 = 0$ from the fact that V is an associative algebra and M a V-module. Figure 11 illustrates the geometric meaning of the terms in the Hochschild differential of the 3-cochain, a, namely.

$$da = m \circ_1 a - a \circ_1 m + a \circ_2 m - a \circ_3 m + m \circ_2 a$$

We view da as being associated to the canonical element in $\mathcal{A}(4)$ and the terms in the differential denote various decompositions of the canonical element in $\mathcal{A}(4)$ into the canonical element in $\mathcal{A}(3)$ and $\mathcal{A}(2)$. The multiplication operation m is associated with the canonical element in $\mathcal{A}(2)$ and a is associated with the canonical elements in $\mathcal{A}(3)$ which are composed as indicated.

We have just shown the following.

Proposition 6. *Let V be an associative algebra (an $\mathcal{A}$-algebra) and M a V-module then the associated cochain complex, $(C_{\mathcal{A}}^\bullet(V, M), d)$, is the Hochschild complex of V with values in M. Furthermore, the inequivalent, infinitesimal deformations of V are classified by elements in $H_{\mathcal{A}}^2(V, V)$.*

3. The little intervals operad. A geometric operad which is well-known from homotopy theory (see [21]) is the *little intervals operad,* $\mathcal{D} = \{\, \mathcal{D}(n) \,\}_{n \geq 1}$. $\mathcal{D}(n)$ is the space of configurations of n distinct, ordered embeddings of I into I where each embedding is a composition of a dilation (multiplication by a positive real number) and a translation (addition of a real number), where the embedded intervals are allowed to intersect at their boundaries. S_n acts on $\mathcal{D}(n)$ by reordering the n maps and the composition maps $\mathcal{D}(n) \times \mathcal{D}(n') \to \mathcal{D}(n+n'-1)$ which take $(\Sigma, \Sigma') \mapsto \Sigma \circ_i \Sigma'$ are given by taking the large interval of Σ' and identifying the large interval with the i-th interval in Σ after shrinking it down to size via a dilation. It will prove useful to construct another realization of this operad.

Let $\mathbb{RP}^1$ inherit an orientation from $\mathbb{R}^2$ so that it is diffeomorphic to an oriented circle. The homogeneous coordinates on $\mathbb{RP}^1$ gives rise to a standard coordinate, x, which identifies $\mathbb{RP}^1$ with $\mathbb{R} \cup \{\, \infty \,\}$. Consider the moduli space of distinct, ordered, real analytic, orientation preserving embeddings $f_i : I \to \mathbb{RP}^1$ for $i = 1, \ldots, n+1$ whose images are mutually disjoint except, possibly, at their boundaries and identify any two such configurations if they are related by an orientation preserving real projective transformation, *i.e.* an element of $\mathrm{PSL}(2, \mathbb{R})$ which acts on $\mathbb{RP}^1$ via

$$x \mapsto \frac{ax+b}{cx+d}, \ \forall\, a, b, c, d \in \mathbb{R} \text{ satisfying } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1.$$

The resulting space, $\mathcal{G}(n)$, for all $n \geq 1$, forms an operad. The permutation group acts by permuting the ordering on the first n maps. The composition maps $\mathcal{G}(n) \times \mathcal{G}(n') \to \mathcal{G}(n + n' - 1)$ which takes $(\Sigma, \Sigma') \mapsto \Sigma \circ_i \Sigma'$ are obtained by cutting out the $(n'+1)$st interval of Σ' and the ith interval of Σ and sewing them together along their boundaries in an orientation preserving fashion. This operad is a real analytic analog of the operad governing conformal field theory. This operad has a suboperad which is isomorphic to the little intervals operad.

Consider an element $(f_1, \ldots, f_n)$ of $\mathcal{D}(n)$ and identify the large interval in an orientation preserving way with the unit interval in the standard coordinate about $x = 0$ in $\mathbb{RP}^1$ and let $f_{n+1} : I \to \mathbb{RP}^1$ be an orientation preserving identification of I with the unit interval in the standard coordinate about $x = \infty$. Let $\mathcal{B}(n)$ consist of those points in $\mathcal{G}(n)$ which arise from configurations of $(n+1)$ distinct, ordered orientation preserving embeddings of I into $\mathbb{RP}^1$ of the type just described. The operad $\mathcal{B}$, which is isomorphic to the little intervals operad, is a suboperad of $\mathcal{G}$. The operad $\mathcal{A}$ shares many features in common with $\mathcal{B}$ (in fact, $\mathcal{A}$ may be regarded as a quotient of the space of elements in $\mathcal{B}$ arising from nonintersecting intervals) but $\mathcal{B}$ has a much richer geometric structure than $\mathcal{A}$. $\mathcal{B}(n)$ is a $2n$ dimensional manifold with corners with $n!$ components where the components are permutated into one another by the free action of S_n, *i.e.* $\mathcal{B}(n) \to \mathcal{B}(n)/S_n$ is a trivial S_n bundle. However, there is a canonical section of this bundle provided by the orientation of $\mathbb{RP}^1$ in analogy to $\mathcal{A}$, *i.e.* assign to each orbit of S_n in $\mathcal{B}(n)$ the unique point consisting of maps whose

ordering increases as one proceeds around $\mathbb{RP}^1$ as prescribed by its orientation. Let $\mathcal{B}'(n)$ denote the image of the canonical section in $\mathcal{B}(n)$ which we shall identify with $\mathcal{B}(n)/S_n$. $\mathcal{B}'$ is closed under the composition maps in analogy with the associative case. For convenience, we shall denote $\mathcal{B}(1) = \mathcal{B}'(1)$ by $\mathcal{K}$ which is a Lie semigroup with unit e.

Just as in the associative operad, the operad $\mathcal{B}$ is quadratic or, more precisely, the linear operad generated by $\mathcal{B}$ is quadratic. All elements in $\mathcal{B}(n)$ are obtained from elements in $\mathcal{K}$ and $\mathcal{B}(2)$ by compositions and the action of the permutation group. However, unlike $\mathcal{A}$, $\mathcal{K}$ contains more than one element – it is a two dimensional manifold with corners. The composition maps endow $\mathcal{B}(n)$ with the structure of a $(\mathcal{K}, \mathcal{K}^n)$-space. $\mathcal{B}$-algebras are determined not only by elements in $\mathcal{B}(2)$ but also by nontrivial elements in $\mathcal{K}$. To be more specific, let Σ be in a point in $\mathcal{B}'(3)$ then, in analogy with equation 6 and figure 10 for the associative case, there exist elements $\Sigma_1, \widetilde{\Sigma}_1, \Sigma_2$, and $\widetilde{\Sigma}_2$ in $\mathcal{B}'(2)$ such that

$$(12) \qquad\qquad \Sigma = \widetilde{\Sigma}_1 \circ_1 \Sigma_1 = \widetilde{\Sigma}_2 \circ_2 \Sigma_2$$

where Σ_i is an element in $\mathcal{B}'(2)$ which can be represented by a circle containing intervals i and $(i+1)$ from the original circle Σ while $\widetilde{\Sigma}_i$ contains the remaining two intervals of the original Σ. However, unlike the associative operad, the decomposition of Σ into Σ_i and $\widetilde{\Sigma}_i$ is not unique because there are nontrivial elements in $\mathcal{K}$. Choose $i = 1, 2$ and Σ in $\mathcal{B}'(3)$ and let $W_i(\Sigma)$ be the set of all pairs $(\widetilde{\Sigma}_i, \Sigma_i)$ in $\mathcal{B}'(2) \times \mathcal{B}'(2)$ such that $\Sigma = \widetilde{\Sigma}_i \circ_i \Sigma_i$. Define an equivalence relation on $W_i(\Sigma)$ by demanding that the relations be generated by $(\Sigma, k \circ_1 \Sigma') \sim (\Sigma \circ_i k, \Sigma')$ for all k in $\mathcal{K}$. In this case, $W_i(\Sigma)/\sim$ consists of one element which means that up to the appropriate action of $\mathcal{K}$, there is a unique way to decompose a given element in $\mathcal{B}'(3)$ into a pair of elements in $\mathcal{B}'(2)$ for a given i. The relations of the operad $\mathcal{B}$ are generated by the free action of S_3 upon equation 12.

4. Deformations of algebras over the little intervals operad.

We shall now obtain the cohomology associated to a $\mathcal{B}$-algebra by studying the infinitesimal deformations of algebras over it.

We say that V is a $\mathcal{B}$-algebra if V is a topological vector space with a smooth morphism of operads $\mu : \mathcal{B} \to \mathcal{E}nd_V$. In particular, a $\mathcal{B}$-algebra V is a $\mathcal{K}$-module with binary operations parametrised by $\mathcal{B}(2)$ which obey quadratic relations. The deformation problem that we will consider is to characterize infinitesimal inequivalent deformations of a given $\mathcal{B}$-algebra V keeping the $\mathcal{K}$-module structure on V fixed.

As in the associative case, the equivariance under the permutation group means that μ is completely determined by its restriction to $\mathcal{B}'$. Let $\rho : \mathcal{K} \to \mathcal{E}nd_V(1)$ (denoted by $k \mapsto \rho_k$) be the restriction of μ to $\mathcal{K}$ and $m : \mathcal{B}'(2) \to \mathcal{E}nd_V(2)$ (denoted by $\Sigma \mapsto m_\Sigma$) be the restriction of μ to $\mathcal{B}'(2)$. ρ makes V into a $\mathcal{K}$-module.

Because μ is quadratic, ρ and m completely determine μ. The operations m and ρ must satisfy, for all k, k' in $\mathcal{K}$, and Σ in $\mathcal{B}'(3)$, Σ_i, $\widetilde{\Sigma}_i$, Σ' in $\mathcal{B}'(2)$, equation 12 and

(1) ($\mathcal{K}$-module structure of V) $\rho_{k \circ_1 k'} = \rho_k \circ_1 \rho_{k'}$,
(2) (($\mathcal{K}, \mathcal{K}^2$)-equivariance of $\mathcal{B}'(3)$)
    (a) $m_{k \circ_1 \Sigma'} = \rho_k \circ_1 m_{\Sigma'}$
    (b) $m_{\Sigma' \circ_1 k} = m_{\Sigma'} \circ_1 \rho_k$,
    (c) $m_{\Sigma' \circ_2 k} = m_{\Sigma'} \circ_2 \rho_k$, and
(3) (associativity) $m_{\widetilde{\Sigma}_1} \circ_1 m_{\Sigma_1} = m_{\widetilde{\Sigma}_2} \circ_2 m_{\Sigma_2}$.

Consider a smooth map $\alpha : \mathcal{B}'(2) \to \mathcal{E}nd_V(2)$ which infinitesimally deforms m to $m' := m + t\alpha$ for small t. Replacing m by m' in the associativity condition 3 and keeping terms first order in t and putting all terms to one side of the equation then one finds that α must be ($\mathcal{K}, \mathcal{K}^2$)-equivariant and satisfy

$$(13) \qquad m_{\widetilde{\Sigma}_2} \circ_2 \alpha_{\Sigma_2} - \alpha_{\widetilde{\Sigma}_1} \circ_1 m_{\Sigma_1} + \alpha_{\widetilde{\Sigma}_2} \circ_2 m_{\Sigma_2} - m_{\widetilde{\Sigma}_1} \circ_1 \alpha_{\Sigma_1} = 0$$

for all Σ in $\mathcal{B}'(3)$, Σ_i, $\widetilde{\Sigma}_i$ in $\mathcal{B}'(2)$ satisfying equation 12. This is the 2-cocycle condition which is analogous to equation 8 in Hochschild cohomology. Suppose that the infinitesimal deformation α, of first order in t, gives rise to a trivial deformation. This occurs if there exists a map β in $\mathcal{E}nd_V(1)$ commuting with the $\mathcal{K}$-module structure such that $\Phi := \mathrm{id}_V - t\beta$ satisfying

$$(14) \qquad m'_\Sigma \circ (\Phi \otimes \Phi) = \Phi \circ m_\Sigma$$

for all Σ in $\mathcal{B}'(2)$. Plugging in $m' = m + t\alpha$ and $\Phi = \mathrm{id}_V - t\beta$ and keeping terms up to first order in t, we obtain

$$(15) \qquad \alpha_\Sigma = m_\Sigma \circ_2 \beta - \beta \circ_1 m_\Sigma + m \circ_1 \beta.$$

Equation 13 is the condition that α is a two cocycle and equation 15 is the condition that α is a two coboundary in the cohomology of a $\mathcal{B}$-algebra V with values in itself. Let M be a V-module then for $n \geq 1$, the space of n-cochains of $\mathcal{B}$-cohomology of V with values in a V-module M $C^n_\mathcal{B}(V, M)$, is defined to be the set of all smooth maps $\alpha : \mathcal{B}'(n) \to \mathrm{Hom}(V^{\otimes n}, M)$ which are ($\mathcal{K}, \mathcal{K}^n$) equivariant. We also define $C^0_\mathcal{B}(V, M) := M$. The differential $d : C^n_\mathcal{B}(V, M) \to C^{n+1}_\mathcal{B}(V, M)$, as suggested by equations 13 and 15, is defined by the formula, for $n \geq 2$,

$$(16) \qquad (d\alpha)_\Sigma = m_{\Sigma_0} \circ_2 \alpha_{\widetilde{\Sigma}_0} + \sum_{i=1}^n (-1)^n \alpha_{\widetilde{\Sigma}_i} \circ_i m_{\Sigma_i} + (-1)^{n+1} m_{\Sigma_{n+1}} \circ_1 \alpha_{\widetilde{\Sigma}_{n+1}}$$

where Σ is an element of $\mathcal{B}'(n+1)$, Σ_i are elements of $\mathcal{B}'(2)$, Σ_i are elements of $\mathcal{B}'(n-1)$, such that for all $i = 1, \ldots, n$,

$$(17) \qquad \Sigma = \widetilde{\Sigma}_i \circ_i \Sigma_i = \Sigma_0 \circ_2 \widetilde{\Sigma}_0 = \Sigma_{n+1} \circ_1 \widetilde{\Sigma}_{n+1}.$$

The expression for the differential is well-defined because of the $(\mathcal{K}, \mathcal{K}^n)$-equivariance of α. The differential of an element β in $C^1_{\mathcal{B}}(V, M)$ is given by

$$(18) \qquad (d\beta)_\Sigma = m_\Sigma \circ_2 \beta_e - \beta_e \circ_1 m_\Sigma + m_\Sigma \circ_1 \beta_e$$

for all Σ in $\mathcal{B}(2)$. Finally, the differential of an element γ in $C^0_{\mathcal{B}}(V, M)$ is given by

$$(19) \qquad (d\gamma)_\Sigma = m_\Sigma \circ_2 \gamma - m_\Sigma \circ_1 \gamma$$

for all Σ in $\mathcal{K}$.

Notice that if β' belongs to $C^1_{\mathcal{B}}(V, M)$ then $(\mathcal{K}, \mathcal{K})$-equivariance means that

$$(20) \qquad \beta'_k = \rho_k \circ_1 \beta'_e = \beta'_e \circ_1 \rho_k$$

for all k in $\mathcal{K}$ so β' is completely determined by β'_e which is an element of $\mathrm{Hom}(V, M)$. It is this $\beta := \beta'_e$ which appears in equation 15.

It is straightforward to verify that $d^2 = 0$ from the associativity condition 12. Figure 11, once again, graphically demonstrates the meaning of the various terms in the differential of a 3-cochain, a, where the Σ_i's correspond to the circles about m and $\widetilde{\Sigma}_i$ correspond to the circles about a. We have just shown the following.

Theorem 7. *Let $\mathcal{B}$ be the little intervals operad, V a $\mathcal{B}$-algebra, and M a V-module. The associated cochain complex is $(C_{\mathcal{B}}(V, M), d)$. Furthermore, inequivalent infinitesimal deformations of V as a $\mathcal{B}$-algebra are classified by its second cohomology, $H^2_{\mathcal{B}}(V, V)$.*

In the special case where V is a $\mathcal{B}$-algebra with a trivial $\mathcal{K}$-module structure then notice that the $(\mathcal{K}, \mathcal{K}^2)$ equivariance of m implies that V is nothing more than an associative algebra and the complex $C^\bullet_{\mathcal{B}}(V, M)$ can be identified with the Hochschild complex of V with values in M.

5. A complex of vertex operator algebras. The little intervals operad, $\mathcal{B}$, embeds into the operad which governs CFT's. Therefore, any CFT is a $\mathcal{B}$-algebra. In this section, we shall write down, explicitly, the complex described in the previous section in the context of vertex operator algebras. This gives rise to a complex written purely in terms of vertex operator algebras which closely resembles Hochschild cohomology.

There is a morphism of operads $\mathcal{B} \hookrightarrow \mathcal{P}$ defined as follows. The canonical inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{C}^2$ induces an inclusion $\mathbb{RP}^1 \hookrightarrow \mathbb{CP}^1$. Given any element Σ in $\mathcal{B}(n)$ represented by embedded intervals, $(f_1, \ldots, f_{n+1})$, in $\mathbb{RP}^1$, then each f_i can be naturally regarded as an embedding of I into $\mathbb{CP}^1$ which, by analytic continuation, becomes a biholomorphic embedding of D into $\mathbb{CP}^1$. The resulting Riemann sphere with holomorphically embedded disks is an element in $\mathcal{P}(n)$. Therefore, if V is a CFT then it is necessarily a $\mathcal{B}$-algebra and its infinitesimal deformations as a $\mathcal{B}$-algebra are governed by the complex defined in the previous section.

Elements in $\mathcal{B}(2)$ are specified by the inclusion maps (f_1, f_2) where $f_1(x) = a_1 x + z_1$ and $f_2(x) = a_2 x + z_2$ for some positive numbers a_1 and a_2 (dilations) as well as real numbers z_1 and z_2 (translations). We shall denote such an element of $\mathcal{B}'(2)$ by (z_1, a_1, z_2, a_2). If V arises from a vertex operator algebra then the multiplication operation $m : \mathcal{B}'(2) \to \mathcal{E}nd_V(2)$ is given by, for all v_1 and v_2 in V,

$$(21) \quad m_{(z_1,a_1,z_2,a_2)}(v_1 \otimes v_2) := \begin{cases} Y((a_1)^{-L_0} v_1, z_1) Y((a_2)^{-L_0} v_2, z_2) \Lambda, & \text{if } |z_1| > |z_2| \\ Y((a_2)^{-L_0} v_2, z_2) Y((a_1)^{-L_0} v_1, z_1) \Lambda, & \text{if } |z_2| > |z_1| \end{cases}$$

and everywhere else m is defined by analytic continuation. L_0 is the usual element in the Virasoro algebra and Λ is the vacuum vector, the image of the element in $\mathcal{P}(0)$ with the standard chart about ∞. This can be used to explicitly write down the differential in $C_{\mathcal{B}}^\bullet$. An element α in $C_{\mathcal{B}}^n(V, M)$ can now be written as $\alpha_{(z_1, a_1, \ldots, z_n a_n)}$ which must be $(\mathcal{K}, \mathcal{K}^n)$ equivariant, using similar coordinates to the above.

Enlarge the spaces $\mathcal{G}(n)$ to a space $\widetilde{\mathcal{G}}(n)$ which is the moduli space of $(n+1)$ distinct, orientation preserving, real analytic coordinates in $\mathbb{RP}^1$ such that no two coordinates have centers which coincide and where any two such are identified if they are related by an action of $\mathrm{PSL}(2, \mathbb{R})$. The collection $\widecheck{\mathcal{G}} := \{\widetilde{\mathcal{G}}(n)\}_{n \geq 1}$ forms a partial operad where the composition maps are given by the one dimensional analogue of sewing in $\widetilde{\mathcal{P}}$. Clearly, $\mathcal{G}$ is a suboperad of the partial operad $\widetilde{\mathcal{G}}$.

There is an important partial suboperad of $\widetilde{\mathcal{G}}$, $\mathcal{C}$, where $\mathcal{C}(n)$ is diffeomorphic to the configuration space of n points on $\mathbb{R}$. That is, let $\mathcal{C}(n)$ consist of $(n+1)$ distinct, ordered orientation preserving embeddings, $(f_1, \ldots, f_{n+1})$, of I into $\mathbb{RP}^1$ where for all $i = 1, \ldots, n$, f_i is embedded into $\mathbb{R}$ by translations only and f_{n+1} is the standard unit interval about ∞ so that no two have centers which coincide. $\mathcal{C} := \{\mathcal{C}(n)\}_{n \geq 1}$ forms a partial operad which is quadratic in the sense that every element in $\mathcal{C}(n)$ can be obtained from compositions of elements in $\mathcal{C}(2)$ and the action of the permutation group. In fact, the cohomology governing deformations of a $\mathcal{C}$-algebra naturally give rise to the a complex for the cohomology of a $\mathcal{C}$-algebra, V, with values in a the V-module, M, $d : C_{\mathcal{C}}^n(V, M) \to C_{\mathcal{C}}^{n+1}(V, M)$ where n-cochains consist of smooth, $(\mathcal{C}(1)^n, \mathcal{C}(1))$-equivariant maps from $\mathcal{C}'(n) \to \mathrm{Hom}(V^n, M)$ where $\mathcal{C}'(n)$ is the canonical component of $\mathcal{C}(n)$. If α is an element in $C_{\mathcal{C}}^n$ and $(z_1, \ldots, z_n)$ is a point in $\mathcal{C}'(n)$ then the differential is given by

$$(22) \quad (d\alpha)_{(z_1,\ldots,z_{n+1})} = m_{(z_1,0)} \circ_2 \alpha_{(z_2,\ldots,z_{n+1})}$$

$$+ \sum_{i=1}^n (-1)^i \alpha_{(z_1,\ldots,z_i, z_{i+2}, \ldots z_{n+1})} \circ_i m_{(0, z_{i+1} - z_i)} + (-1)^{n+1} m_{(0, z_{n+1})} \circ_1 \alpha_{(z_1,\ldots,z_n)}$$

where m denotes the binary multiplication on V or the module action of V on M as it appropriate. We have shown the following.

Theorem 8. *Let V be a $\mathcal{C}$-algebra and M, a V-module then $(C_{\mathcal{C}}^{\bullet}(V, M), d)$ is the associated cochain complex which governs deformations of $\mathcal{C}$-algebras.*

In the case where V is a vertex operator algebra and v_1, v_2 belong to V, m is given by

$$(23) \qquad m_{(z_1, z_2)}(v_1 \otimes v_2) := \begin{cases} Y(v_1, z_1)Y(v_2, z_2)\Lambda, & \text{if } |z_1| > |z_2| \\ Y(v_2, z_2)Y(v_1, z_1)\Lambda, & \text{if } |z_2| > |z_1| \end{cases}$$

and is defined by analytic continuation elsewhere.

In the definition of a vertex operator algebra, the most important axiom is the so-called associativity axiom. Formally deforming the map Y infinitesimally, we are led to an expression that is precisely that which is written above.

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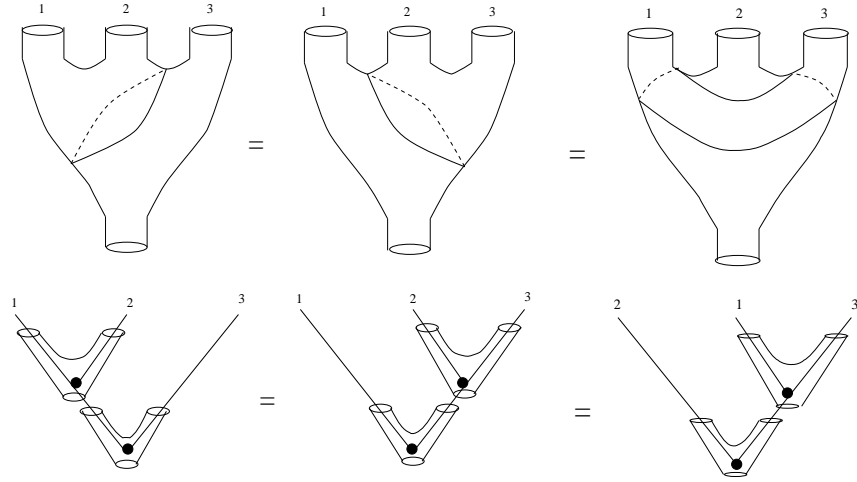   *E-mail address*: voronov@math.upenn.edu

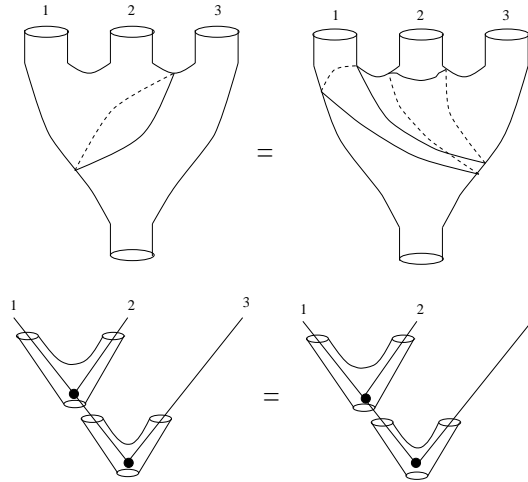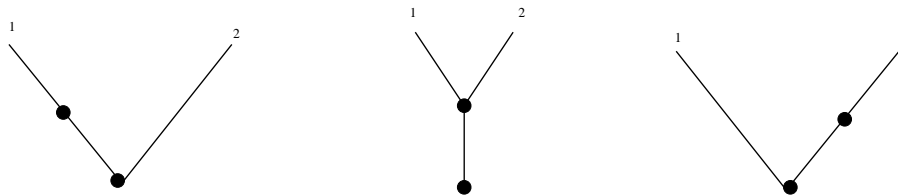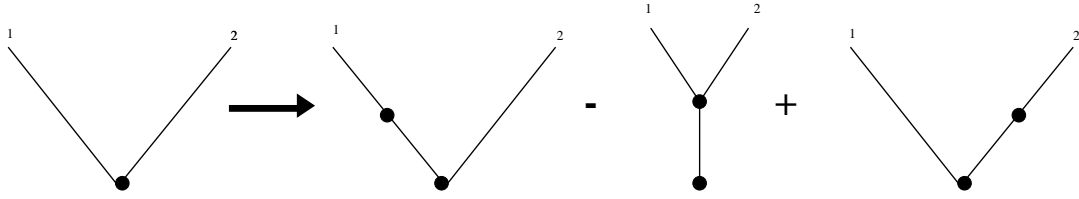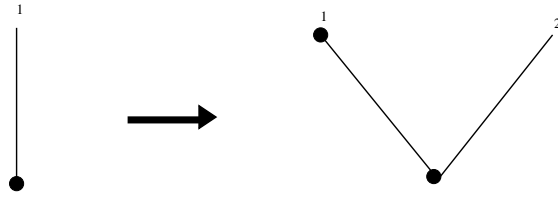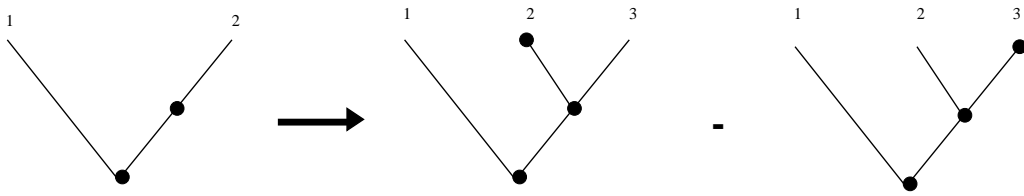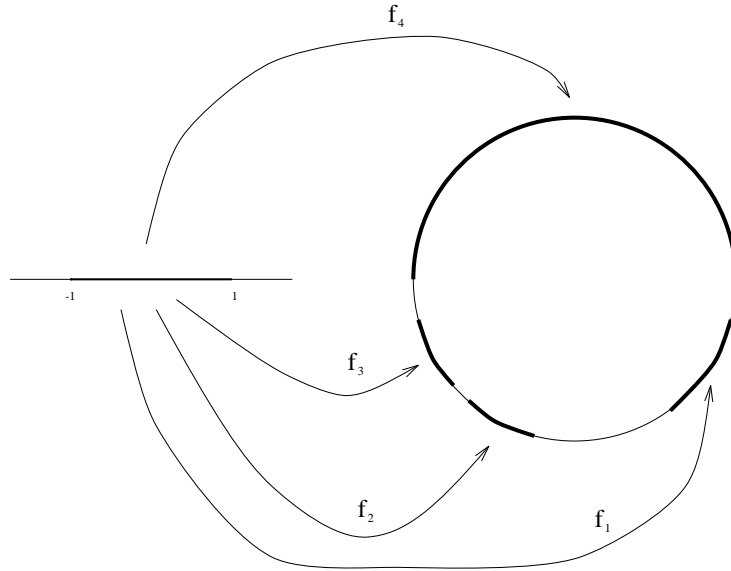FIGURE 3. 4-Point Relations Between Different Trees

FIGURE 4. A 4-Point Relation Between the Same Trees

FIGURE 5. Trees Indexing $(E \otimes_K K)^{\oplus 3}$

FIGURE 6. Action of δ_1

FIGURE 7. Action of $s : K \rightarrow E \otimes_K C$

FIGURE 8. Action of δ_0 on a Component of $(E \otimes_K K)^{\oplus 3}$

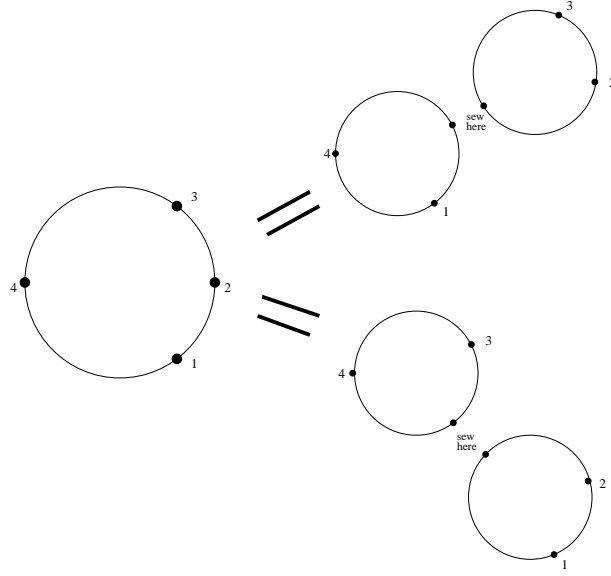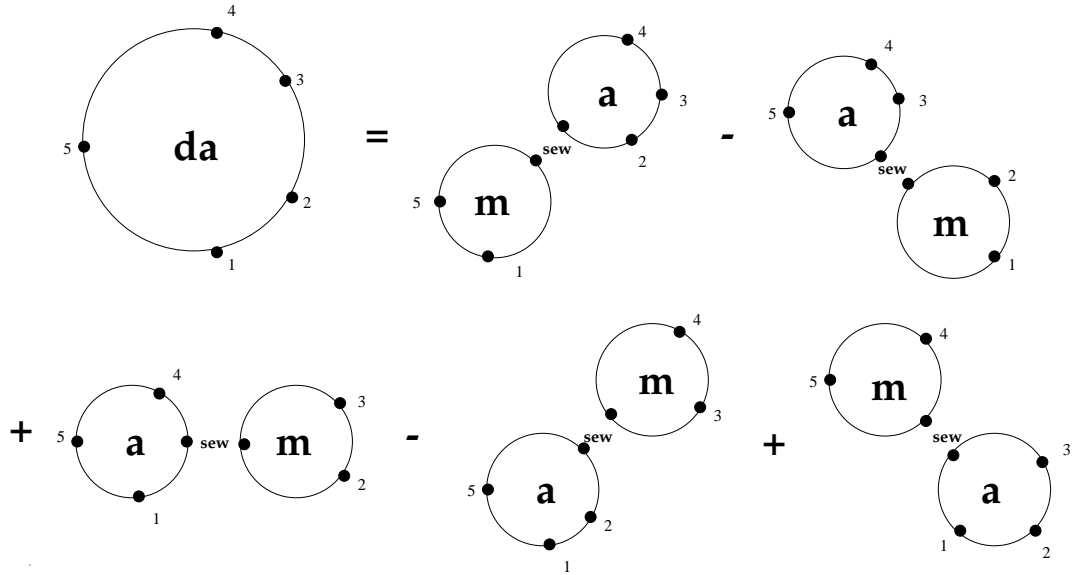FIGURE 9. The Canonical Element in A(3)

FIGURE 10. The Relations of $\mathcal{A}$ Involving the Canonical Elements in A(3)

FIGURE 11. The Differential of a 3-cocycle